Chapter 6: Solutions to Problems 1, 2, 4, 6-8

1. $t = -2N \ln\left(\frac{H_t}{H_0}\right)$

$H_0(0.05) = H_t$

$t = -2(10)\ln(0.05) = 59.9$ → 60 generations

$t = -2(100)\ln(0.05) = 599.1$ → 599 generations

2. $x_{ij} = \frac{(2N)!}{(2N-i)!i!}\left(1-\frac{j}{2N}\right)^{2N-i}\left(\frac{j}{2N}\right)^{i}$

Prob (state 0 →0) = 1

Prob (state 0 →1, 2, 3, 4) = 0

Prob (state 1 → 0) $= x_{01} = \frac{4!}{4!0!}\left(1-\frac{1}{4}\right)^{4}\left(\frac{1}{4}\right)^{0} = 0.3164$

Prob (state 1 → 1) $= x_{11} = \frac{4!}{3!1!}\left(1-\frac{1}{4}\right)^{3}\left(\frac{1}{4}\right)^{1} = 0.4219$

Prob (state 1 → 2) $= x_{21} = \frac{4!}{2!2!}\left(1-\frac{1}{4}\right)^{2}\left(\frac{1}{4}\right)^{2} = 0.2109$

Prob (state 1 → 3) $= x_{31} = \frac{4!}{1!1!}\left(1-\frac{1}{4}\right)^{1}\left(\frac{1}{4}\right)^{3} = 0.0469$

Prob (state 1 → 4) $= x_{41} = \frac{4!}{0!4!}\left(1-\frac{1}{4}\right)^{0}\left(\frac{1}{4}\right)^{4} = 0.0039$

Prob (state 2 → 0) $= x_{02} = \frac{4!}{4!0!}\left(1-\frac{2}{4}\right)^{4}\left(\frac{2}{4}\right)^{0} = 0.0625$

Prob (state 2 →1) $= x_{12} = \frac{4!}{3!1!}\left(1-\frac{2}{4}\right)^{3}\left(\frac{2}{4}\right)^{1} = 0.25$

Prob (state 2→ 2) $= x_{22} = \frac{4!}{2!2!}\left(1-\frac{2}{4}\right)^{2}\left(\frac{2}{4}\right)^{2} = 0.375$

Prob (state 2→ 3) $= x_{32} = \frac{4!}{1!3!}\left(1-\frac{2}{4}\right)^{1}\left(\frac{2}{4}\right)^{3} = 0.25$

Prob (state 2→ 4) $= x_{42} = \frac{4!}{0!4!}\left(1-\frac{2}{4}\right)^{0}\left(\frac{2}{4}\right)^{4} = 0.0625$

Prob (state 3→ 0) $= x_{03} = \frac{4!}{4!0!}\left(1-\frac{3}{4}\right)^{4}\left(\frac{3}{4}\right)^{0} = 0.0039$

Prob (state 3→ 1) $= x_{13} = \frac{4!}{3!1!}\left(1-\frac{3}{4}\right)^{3}\left(\frac{3}{4}\right)^{1} = 0.0469$

Prob (state 3→ 2) = $x_{23} = \frac{4!}{2!2!}\left(1-\frac{3}{4}\right)^2\left(\frac{3}{4}\right)^2$ = 0.2109

Prob (state 3→ 3) = $x_{33} = \frac{4!}{1!3!}\left(1-\frac{3}{4}\right)^1\left(\frac{3}{4}\right)^3$ = 0.4219

Prob (state 3→ 4) = $x_{43} = \frac{4!}{0!4!}\left(1-\frac{3}{4}\right)^0\left(\frac{3}{4}\right)^4$ = 0.3164

Prob (state 4→ 0, 1, 2, 3) =0

Prob (state 4→ 4) = 1

	Generation t				
t+1	0	1	2	3	4
0	1	0.3164	0.0625	0.0039	0
1	0	0.4219	0.25	0.0469	0
2	0	0.2109	0.375	0.2109	0
3	0	0.0469	0.25	0.4219	0
4	0	0.0039	0.0625	0.3164	1

$$X = \begin{vmatrix} 1 & 0.3164 & 0.0625 & 0.0039 & 0 \\ 0 & 0.4219 & 0.25 & 0.0469 & 0 \\ 0 & 0.2109 & 0.375 & 0.2109 & 0 \\ 0 & 0.0469 & 0.25 & 0.4219 & 0 \\ 0 & 0.0039 & 0.0625 & 0.3164 & 1 \end{vmatrix} \qquad Y = \begin{vmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{vmatrix}$$

$$Y' = \begin{vmatrix} 1(0.2)+0.3164(0.2)+0.0625(0.2)+0.0036(0.2)+0(0.2) \\ 0(0.2)+0.4219(0.2)+0.25(0.2)+0.0469(0.2)+0(0.2) \\ 0(0.2)+0.2109(0.2)+0.375(0.2)+0.2109(0.2)+0(0.2) \\ 0(0.2)+0.0469(0.2)+0.25(0.2)+0.4219(0.2)+0(0.2) \\ 0(0.2)+0.0039(0.2)+0.0625(0.2)+0.3164(0.2)+1(0.2) \end{vmatrix} = \begin{vmatrix} 0.277 \\ 0.144 \\ 0.159 \\ 0.144 \\ 0.277 \end{vmatrix}$$

$$H_1 = 2\sum_{j=0}^{2N}\left(1-\frac{j}{2N}\right)\left(\frac{j}{2N}\right)y_j$$

$$H_1 = 2\left[\begin{array}{l} \left(1-\frac{0}{4}\right)\left(\frac{0}{4}\right)0.277+\left(1-\frac{1}{4}\right)\left(\frac{1}{4}\right)0.144+\left(1-\frac{2}{4}\right)\left(\frac{2}{4}\right)0.159 \\ +\left(1-\frac{3}{4}\right)\left(\frac{1}{4}\right)0.144+\left(1-\frac{4}{4}\right)\left(\frac{4}{4}\right)0.277 \end{array}\right] = 0.1875$$

$$Y' = \begin{vmatrix} 1(0.277) + 0.3164(0.144) + 0.0625(0.159) + 0.0036(0.144) + 0(0.277) \\ 0(0.277) + 0.4219(0.144) + 0.25(0.159) + 0.0469(0.144) + 0(0.277) \\ 0(0.277) + 0.2109(0.144) + 0.375(0.159) + 0.2109(0.144) + 0(0.277) \\ 0(0.277) + 0.0469(0.144) + 0.25(0.159) + 0.4219(0.144) + 0(0.277) \\ 0(0.277) + 0.0039(0.144) + 0.0625(0.159) + 0.3164(0.144) + 1(0.277) \end{vmatrix} = \begin{vmatrix} 0.333 \\ 0.107 \\ 0.120 \\ 0.107 \\ 0.333 \end{vmatrix}$$

$$H_2 = 2\begin{bmatrix} \left(1-\frac{0}{4}\right)\left(\frac{0}{4}\right)0.333 + \left(1-\frac{1}{4}\right)\left(\frac{1}{4}\right)0.107 + \left(1-\frac{2}{4}\right)\left(\frac{2}{4}\right)0.120 \\ + \left(1-\frac{3}{4}\right)\left(\frac{1}{4}\right)0.107 + \left(1-\frac{4}{4}\right)\left(\frac{4}{4}\right)0.333 \end{bmatrix} = 0.140$$

4. $R = 1 - \sum p_i^{2N}$

if $p = q = 0.5$

$R = 1 - \left[(0.5)^{(2)(4)} + (0.5)^{(2)(4)}\right] = 0.992$

if $A_1 .. A_{10} = 0.1$

$R = 1 - \left[(10)(0.1)^{(2)(4)}\right] = 0.9999$

6. $$N_e = \frac{4N_f}{N_f + 1}$$

$$N_e = \frac{4(5)}{5+1} = 3.33$$

$$N_e = \frac{9N_f N_m}{2N_f + 4N_m}$$

$$N_e = \frac{9(1)(10)}{2(1)+4(10)} = 2.14$$

7. $$N_e = \frac{1}{2\left\{1-\left[\prod_{i=0}^{t-1}\left(1-\frac{1}{2N_{e.i}}\right)\right]^{\frac{1}{t}}\right\}}$$

$$\frac{1}{2\left\{1-\left[\left(1-\frac{1}{(2)(5)}\right)\left(1-\frac{1}{(2)(50)}\right)\left(1-\frac{1}{(2)(10)}\right)\left(1-\frac{1}{(2)(100)}\right)\right]^{\frac{1}{4}}\right\}} = 11.90$$

$$N_e = \frac{t}{\sum \frac{1}{N_i}}$$

$$N_e = \frac{4}{\frac{1}{5}+\frac{1}{50}+\frac{1}{10}+\frac{1}{100}} = 12.12$$

Equation 6.12b which is an approximation given that N_e is not too small is slightly greater than that using 6.12a.

8. $$Ne_f = 100$$

$$Ne_m = 10$$

$$Ne_{mtDNA} = \frac{Ne_f}{2} = \frac{100}{2} = 50$$

$$Ne_Y = \frac{Ne_m}{2} = \frac{10}{2} = 5$$

$$Ne = \frac{4Ne_f Ne_m}{Ne_f + Ne_m} = \frac{4(100)(10)}{100+10} = 36.6$$