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Non-verbal numerical cognition: from reals to integers

C.R. Gallistel and Rochel Gelman

Data on numerical processing by verbal (human) and non-verbal (animal and human) subjects are integrated by the hypothesis that a non-verbal counting process represents discrete (countable) quantities by means of magnitudes with scalar variability. These appear to be identical to the magnitudes that represent continuous (uncountable) quantities such as duration. The magnitudes representing countable quantity are generated by a discrete incrementing process, which defines next magnitudes and yields a discrete ordering. In the case of continuous quantities, the continuous accumulation process does not define next magnitudes, so the ordering is also continuous ('dense'). The magnitudes representing both countable and uncountable quantity are arithmetically combined in, for example, the computation of the income to be expected from a foraging patch. Thus, on the hypothesis presented here, the primitive machinery for arithmetic processing works with real numbers (magnitudes).

The study of numerical estimation and reasoning in non-verbal animals has affected contemporary theories of human numerical cognition, and of its ontogeny and phylogeny[1–7]. According to one emerging synthesis of these findings, the tension between the discrete and the continuous, which has been central to the historical development of mathematical thought, is rooted in the non-verbal foundations of numerical thinking. It is argued that these foundations are common to humans and non-verbal animals. In this view, the non-verbal representatives of number are mental magnitudes (real numbers) with scalar variability. Scalar variability means that the signals encoding these magnitudes are 'noisy'; they vary from trial to trial, with the width of the signal distribution increasing in proportion to its mean. In short, the greater the magnitude, the noisier its representation. These noisy mental magnitudes are arithmetically processed – added, subtracted, multiplied, divided and ordered. Recognition of the importance of arithmetically processed mental magnitudes in the non-verbal representation of number has emerged from a convergence of results from human and animal studies. This is a fruitful area of comparative cognition.

The relationship between integers and magnitudes is asymmetrical: magnitudes (real numbers) can represent integers but integers cannot represent magnitudes. The impossibility of representing magnitudes, such as the lengths of bars, as countable (integer) quantities has been understood since the ancient Greeks proved that there is no unit of length that divides a whole number of times into both the diagonal and the side of a square. Equivalently, the square root of 2 is an irrational number, a number that cannot be expressed as a proportion between countable quantities. By contrast, when one draws a histogram, there is no count that cannot be represented by the length of a bar.

Intuitively, however, the numbers generated by counting seem to be the foundation of mathematical thought. Twentieth-century mathematicians have commonly assumed that mathematics rests on what is intuitively given through verbal counting, a view epitomized in Kronecker's often quoted remark that, 'God made the integers, all else is the work of man' (quoted in Ref. 8, p. 477). 'All else' includes the real numbers, all but a negligible fraction of which are irrational. Irrational numbers can only be defined rigorously as the limits of infinite series of rational numbers, a definition so elusive

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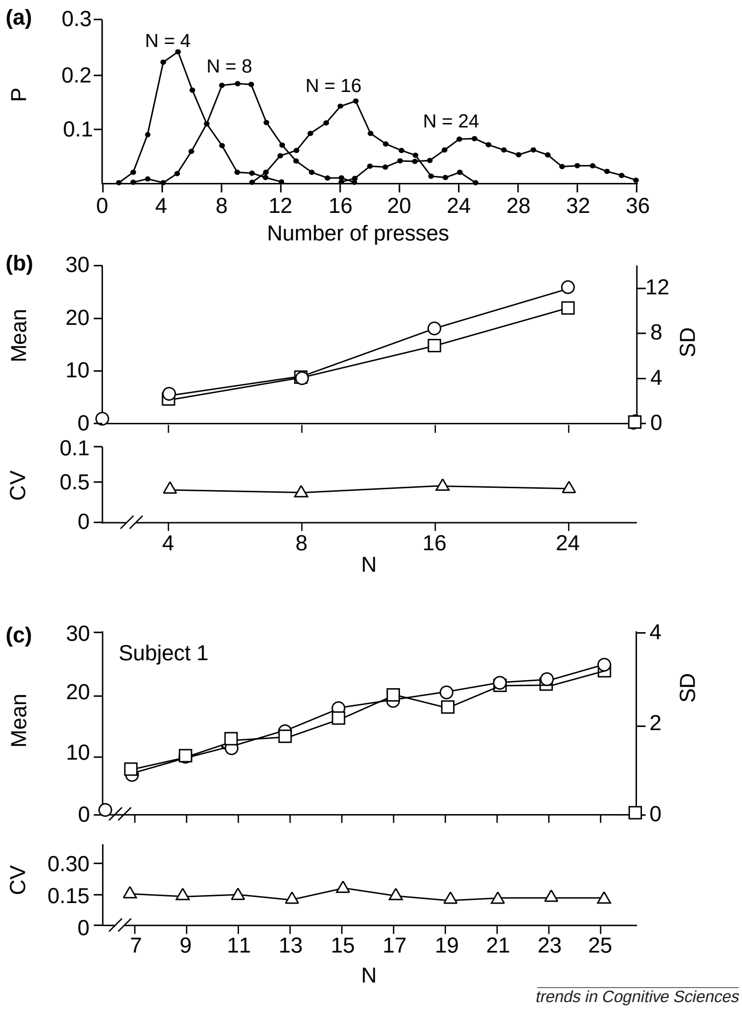

Fig. 1. Experiments showing scalar variability. (a) The probability of breaking off a sequence of lever presses to try the feeder as a function of the number of presses in the sequence and the number required to arm the feeder (N). (Adapted from Ref. 8.) **(b)** The mean (left axis, filled circles) and standard deviation (right axis, filled squares) of the distributions in A, and the coefficient of variation (CV), which is the ratio of the standard deviation to the mean (triangles, lower panel). Note the constancy of the CV. **(c)** The mean (left axis, upper panel) and standard deviation (right axis, upper panel) and the CV (lower panel) of the distributions of number of presses obtained by Whalen *et al.*[19] in experiments with humans using non-verbal counting to make the number of presses specified by a numeral. As with the animal data, the human data show scalar variability. The widths of the distributions of numbers of key presses increase in proportion to the mean number of presses for a given target number, thus the CV is constant.

and abstract that it took more than two thousand years to achieve – an arduously reached pinnacle of mathematical thought. We suggest that the scaling of this pinnacle was a Platonic rediscovery of what the non-verbal brain was doing all along – using arithmetically processed magnitudes to represent both countable and uncountable quantities.

Scalar variability

Objective magnitudes, unlike objective numerosities, cannot be known exactly. This is true for subjective (mental) magnitudes as well, because the scalar noise (mentioned above) in the process of recalling mental magnitudes from memory leads to trial-to-trial variability in the recalled magnitudes. The discovery of scalar variability in the non-verbal representation of numerosity has been a key aspect of the convergence between animal and human studies of numerical cognition. It means that numerosity is never represented exactly in the non-verbal or preverbal mind, with the possible exception of the first three or four numerosities (see Outstanding questions).

Scalar variability in the rat's memory for a target number was evident in the results from an early experiment in which a feeder was silently armed when the rat had made a fixed number of lever presses[9]. Trying the feeder after it was armed, that is, interrupting the infrared beam in front of it, released food; trying it prematurely produced a short but frustrating time out. Rats learned to try the feeder after making approximately the required number of presses. The modal number of presses prior to a try was close to the required number, but in retrospect the most striking aspect of the data was that the width of the distributions increased in proportion to their mode (Figs 1a and 1b). The trial-to-trial variability in the accuracy with which they approximated the target number was proportional to the magnitude of the target, even for numbers as small as four.

Of course, number and duration tend to co-vary in discrimination experiments when the events to be counted occur sequentially. The different possible bases for the discrimination have been experimentally teased apart in several ways[10]. An elegant way to dissociate the two dimensions is a recently developed paradigm in which the basis for a discrimination on a given trial (either the number of flashes or the duration of a sequence of flashes) is not specified until after the sequence of flashes has been presented[11]. The subjects (in this case pigeons) must count and time the sequence, then make opposing choices based either on the flash count or on the duration of the sequence, according to which dimension is subsequently indicated as the relevant dimension for that trial. From this and other work, it appears that laboratory animals simultaneously time and count stimulus sequences[11–17], although there is some disagreement about whether they tend to rely on counting even when they are in a timing task[15] or, conversely, tend to rely on timing even when they are in a counting task[17].

Experiments with humans

At about the time of Platt and Johnson's experiment with rats[9], Moyer and Landauer[18] measured the reaction latencies in a task in which adult human subjects were asked to indicate which of two numerals represented the bigger number, with many repeated trials. The bigger the two numbers ('size effect') and the smaller the difference between them ('distance effect'), the longer it took subjects to choose the numeral representing the bigger number. Moyer and Landauer explained these size and distance effects by assuming that numbers were represented in the brain by magnitudes, which obeyed Weber's law. This law states that the discriminability of two perceived magnitudes (e.g. two weights) is determined by the ratio of the objective magnitudes. In one version of Moyer and Landauer's hypothesis, as two numbers with a fixed difference become larger, the ratio of the signals that represent them (the ratio of their subjective magnitudes) becomes smaller. If signal variability (noise) is proportional to signal strength (scalar variability), then the smaller the ratio of two subjective magnitudes, the greater the overlap in the signal distributions; hence the more difficult it is to discriminate the signal for one number from the signal for the other. The increased difficulty of the discrimination translates into longer reaction times by way of a speed–accuracy trade-off function.

The Moyer and Landauer approach measures a secondary, and probably non-linearly related, consequence of the presumed variability in the strengths of the signals representing numbers. Whalen and his collaborators[19] adapted the procedure of Platt and Johnson (first used by Mechner[20]) to demonstrate directly non-verbal counting and scalar trial-to-trial variability of the target magnitudes in adult humans. They repeatedly presented numerals representing numbers between 7 and 25 on a VDU and asked subjects to press a button as fast as they could until they felt they had made the indicated number of button presses. The subjects were told not to count their button presses, but to do the task 'by feel'. The resulting distributions of numbers of button presses closely resembled the animal data: the modal number of presses increased in proportion to the target number, and so did the standard deviations of the distributions (Fig. 1c).

Several lines of evidence suggest that the subjects did not count subvocally during the experiment. First, they made one button press every 125 ms – more than twice as fast as estimates of the rate of subvocal counting. Second, there was no step at or after 10 in the function relating inter-press interval to the number of button presses. Such a step would be expected if responses were paired with verbal counts, because counting verbally (out loud or subvocally) requires the articulation of two syllables per count when the count exceeds 10, but only one syllable per count below 10 (7 and 12 being the only exceptions to this). Third, according to preliminary reports, similar results are obtained even when subjects repeat the phrase 'Mary had a little lamb' as they make their presses (S. Cordes, R. Gelman, C.R. Gallistel and J. Whalen, unpublished data). This modification of the procedure helps to rule out subvocal counting, insofar as one assumes that subjects cannot subvocally articulate count words at the rate of eight words per second while simultaneously saying out loud an unrelated phrase at the normal speaking rate of two to three words per second.

These results suggest that humans share with non-verbal animals a non-verbal counting process. In this model, this process repeatedly increments a magnitude that represents the current count. This discretely incremented magnitude is compared with the magnitude that represents the target count to determine when the current count has reached the target count (Fig. 2). The scalar variability in the number of counts made for a given target count is evidence that the incrementing process itself is not the primary source of variability. The longer a count, the greater the likelihood of too many or too few increments (miscounts). However, variability due to miscounts should behave like binomial variability: it should increase in proportion to the square root of the target count. In fact, the experimentally observed variability increases in proportion to the target count itself.

These results further suggest that numerate subjects have learned to map from number words and numerals to the magnitudes that the non-verbal counting process generates to represent number. In order to terminate their button pressing at approximately the right count, the subjects in the Whalen *et al.* experiment had to compare the continually

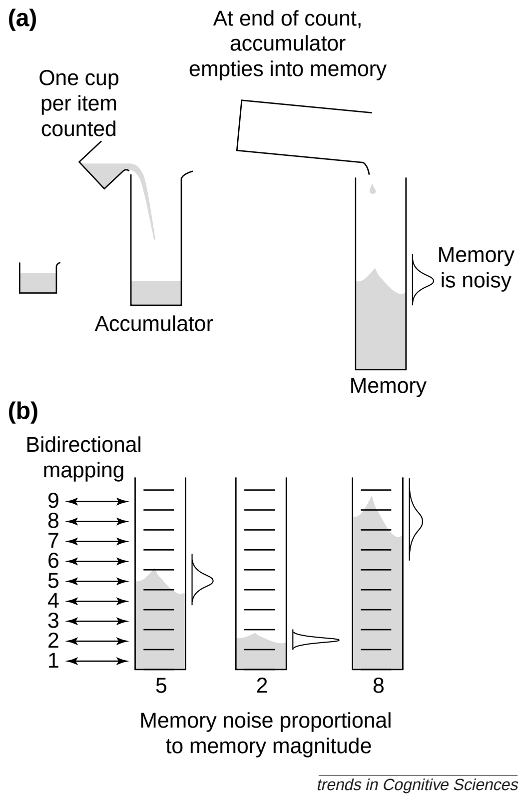

Fig. 2. Cartoon of the accumulator model and the bidirectional mapping hypothesis. This hypothesis is used to explain the results of Whalen *et al.*'s experiment[19]. **(a)** The non-verbal counting process increments the accumulator by one 'cup' for each item counted. The accumulation at the end of a count is read into memory. Magnitudes read from memory to serve as the targets in subsequent counting trials exhibit scalar variability, represented here by the sloshing in the beaker, which introduces noise into readings of the beaker (recalled magnitudes). **(b)** Adult humans have learned decision criteria (rulings on the beaker), which enable them to map from a magnitude to a numeral and from a numeral to a corresponding magnitude (bi-directional mapping hypothesis). Whalen *et al.*[19] argue that the magnitude representation of numerosity and the non-verbal counting process that generates the magnitudes make the formally identical verbal counting process intelligible to the very young child and gives the number words their meanings. (Adapted from Ref. 19.)

incremented magnitude representing the current count with a target magnitude, which was indicated by a numeral[19] (see Fig. 2). A mapping from numerals to magnitudes representing number in the brain was, of course, the essence of the Moyer and Landauer model[18].

Counting and timing

The hypothesis we present here takes as its starting point the accumulator model, which was proposed by Meck and Church[12], as a modification of a model previously proposed by Gibbon[21] to explain interval timing. In this model of timing, there is an accumulator which integrates a steady signal throughout the interval being timed. When the interval terminates, the magnitude in the accumulator represents the duration of the interval. This magnitude is written to memory, to be read from memory when a target magnitude is needed for subsequent comparisons. Meck and Church pointed out that if the steady signal is gated by a pulse former, which pulses once for each item or event to be counted, then the accumulation (integral) at the end of the count represents the number of items or events in the set or series controlling the pulse former. In seminal experiments, Meck and Church showed that numerosity (countable quantity) is represented by magnitudes indistinguishable from those that represent duration (uncountable quantity), with the same constant ratio between the standard deviation and the mode in the distributions of remembered magnitudes[12,22].

Although the counting mechanism just described generates magnitudes (real numbers), it does so by a discrete incrementing process, which defines a next magnitude, just as ordinary counting defines a next integer. By contrast, the timing mechanism does not define next magnitudes. As the duration of a timed interval increases, the timing mechanism generates bigger magnitudes to represent that duration, but it does not pick out a magnitude that is the next magnitude. One way to visualize this is to imagine that, in the counting case, the accumulator is filled one cupful after the next, but in the timing case, the accumulator is filled by a hose, the flow from which is terminated at the end of an arbitrary interval. The distinction between the integers and the reals, between the discrete and the continuous, lies precisely here: integers are discretely ordered and countably infinite, like the levels you get when you fill an (infinitely tall) beaker one cupful at a time. By contrast, the reals are continuously ordered and uncountably infinite, like the levels you get when you fill the beaker with a hose that is 'on' for different, random amounts of time.

The assumption that the preverbal representatives of numerosity are magnitudes with something like scalar variability offers one explanation of the results from numerical discrimination experiments in human infants. Infants discriminate small numerosities with high ratios, such as three versus two and sometimes four versus three but fail to discriminate four versus five. Another explanation of this finding has been that infants and animals estimate small numerosities only by 'subitizing'[23-25]. Subitizing is a frequently hypothesized perceptual process of a kind, which, by assumption, does not involve counting, and which cannot represent more than about four objects at one time. On this account, the failure of infants to discriminate numerosities greater than four implies that the subitizing process provides an initial and very limited representation of numerosity, different from the adult representation[24]. However, it has recently been found that infants do discriminate large numerosities, *provided* that their ratio is large (e.g. 8 versus 16)[26]. This finding is consistent with the hypothesis that the failures of numerical discrimination found in infants are rooted in the noisiness of their non-verbal representation of numerosity rather than in an ontogenetic discontinuity in the mode of numerical representation.

Much of the argument for a subitizing process rests on the claim that there is a discontinuity at about four in the function relating the numerosity of an array to the mean latency with which an adult subject can verbally estimate that numerosity. Whether the experimental data in fact show any such discontinuity has been debated[23,27–30]. A recent and thorough experimental study of the reaction-time distributions for verbal numerosity judgments in adults found no evidence of a discontinuity in any parameter of these distributions[30].

Interrelated questions of discontinuity versus continuity are central to the divergences in current views of numerical cognition and its development (see Outstanding questions). On one view, there is a discontinuity at around four in the adult representation of numerosity. This corresponds to an ontogenetic discontinuity, in which infants can only represent numerosities less than or equal to about four[24]. Accounts that stress the importance of verbal processing in the emergence of the human understanding of number implicitly assume a phylogenetic discontinuity[24,31,32]. The extreme form of the phylogenetic discontinuity hypothesis is the common assumption that animals cannot represent numerosity at all. Our hypothesis, by contrast, is that there is phylogenetic and ontogenetic continuity in numerical processing. We argue that there is a non-verbal representation of numerosity by means of arithmetically processed noisy magnitudes in both non-verbal animals and human infants. We argue further that these magnitudes are identical with the 'semantic magnitude system', which many assume to be addressed by human adults when they process integers[2,33].

The above-mentioned finding that duration and number appear to be represented by mental magnitudes with identical properties suggests that we can generalize important findings about the magnitudes that represent duration to the magnitudes that we assume represent number. One such finding is that the scalar variability (noise) in remembered magnitudes appears to originate within memory itself or in the reading of memory, rather than in the processes for measuring duration or counting number[34]. What this means is that memory signals, like sensory signals, are noisy: the value (signal) obtained from reading the same memory repeatedly varies from reading to reading.

Other important findings that may also generalize concern the arithmetic processing of mental magnitudes, to which we now turn.

Arithmetic reasoning in animals

Countable and uncountable quantity (numerosity and amount, duration, etc.) should be represented with the same kind of symbols (mental magnitudes), because there are many cases in which the two kinds of quantity must be combined arithmetically to determine behaviorally important decision variables. One example might be the amounts of food to be expected per unit of time in different feeding patches. Representing countable and uncountable quantity in fundamentally different ways would be an obstacle to the realization of the requisite combinatorial operations. Indeed, the necessity of representing discrete and continuous quantity with a single coherent number system drove the prolonged effort to create the system of real numbers. It also explains why computers are either analog or digital, but rarely both. You cannot add a voltage (analog representation) to a bit pattern (digital representation). Thus, if you want to process digital signals (e.g. integers) arithmetically in an analog computer, you must convert them to analog form, and vice versa.

This consideration brings us to the evidence for arithmetic processing in animals. Do animals order, add, subtract, multiply and divide the magnitudes that represent both numbers and uncountable quantities like duration and amount? If so, then, on the authors' view, at least, animals reason arithmetically, because the axioms defining these operations define the system of arithmetic. If an unknown combinatorial operation (in the mind or the brain) can be shown – by an examination of its behavioral effects – to exhibit the properties that are unique to one or another mathematical form of combination (addition, multiplication, etc.), then, from a formalist perspective, the unknown operation is an instance of addition or multiplication, and so on.

Ordering

Consider an experiment like that of Platt and Johnson (Fig. 1), where the subject learns to try the feeder only when the number of times it has pressed the lever is greater than or equal to some experimenter-specified value. In all models of these results that we know of, a signal in the mind or brain that is a monotonic function of the number of lever presses is compared with a signal from memory that represents the target value. The response of trying the feeder occurs only when the one signal is greater than the other. Insofar as such an assumption proves inescapable, the results from these experiments are evidence for a thresholding process in the mind or brain. Thresholding has the formal properties of the numerical ordering operation (ordination). Similarly, if, in order to explain the properties of the experimentally obtained distributions of responses, it proves necessary to assume that the quantity that is thresholded is the ratio of the first signal to the second, then these experiments also give evidence for a combinatorial operation in the mind or brain that has the properties of division[35].

These arguments are, however, inferential and theory-dependent. Recently, more direct evidence of ordination has been reported in monkeys and pigeons. Brannon and her collaborators trained rhesus macaques to touch four simultaneously presented arrays in the order of the numbers of items in the arrays[36,37]. The monkeys were trained using arrays with numerosities from 1 to 4, and then tested on arrays with numerosities in the range from 5 to 9. They obtained immediate transfer, which implies that their subjects recognized numerical order in this range. Interestingly, it was not possible to teach monkeys to touch the arrays in an order different from the ordering implied by the numbers of items in the arrays, which implies that numerical order is a salient property of arrays that differ in numerosity. Brannon *et al.*'s data[37] also show the size and distance effects that led Moyer and Landauer to suggest that the ordering of numbers by humans was mediated by a magnitude representation.

In short, the greater the numbers and the smaller the difference between them, the longer it takes both monkeys and human adults to determine their numerical order. This is evidence for a serial counting process of some kind in the determination of the representation of simultaneously presented numerosities. It would be interesting to test for scalar variability or the lack thereof in the representation of simultaneously presented arrays, because miscounts (counting the same item twice or missing an item) would seem to be more likely with simultaneous as opposed to serial presentation. Emmerton and her collaborators[38] taught pigeons to discriminate successively presented visual arrays on the basis of the number of items in each array, with numerosities ranging from 1 to 7. The smaller numerosities were better discriminated, suggesting that the size and distance effects are found in pigeons as well. Note that although the arrays to be discriminated were presented successively, each array constituted a simultaneously presented numerosity.

Numerical order can be the basis of discriminative choice even when the numbers are not actually presented but are indicated by symbols (numerals) whose numerical value has been taught[39]. Thus, ordination on the basis of (symbolically referred to) numerosity is observed even when there is nothing to count.

Adding

Olthof and her collaborators[40] taught two Rhesus macaques to choose the Arabic numeral (range 1–9) indicating the larger of two numerosities or the largest of four numerosities by rewarding them with the number of peanuts indicated by the numeral they chose. When these subjects were presented arrays of numerals (two numerals in one array versus one in the other, or two and two, or three and three), they showed a significant tendency to choose the array whose indicated numbers had the greater sum. This is reminiscent of an earlier result by Boysen and Berntson[41] in which chimpanzees viewed numerals in two different locations, then returned to the starting point and picked the numeral corresponding to the sum of the indicated numbers. Experiments that demonstrate the summation of the numbers indicated by numerals are important because the addition observed is not readily explained by 'counting on', in which a subject adds by counting the items in one array and then continuing the count with the items in the second array. These results show that adding can be observed even when there is nothing to count.

Subtracting

Hauser and his collaborators used a paradigm that evokes prolonged looking by 'violation-of-expectations' to demonstrate that semi-wild, untrained rhesus monkeys were surprised when the results of observed additions to and subtractions from an occluded array were not what was to be expected from arithmetic processing[42]. This is the paradigm that has been used to demonstrate arithmetic reasoning in human infants[1,34,43,44], and it provides another example of methodological convergence in the comparative study of numerical cognition. A potential objection, however, is that because of the very small numerosities used in this paradigm, the results can be explained in terms of the subjects' belief in object permanence[7,24]. Objects do not appear out of, or vanish into, thin air. Subjects can keep track of three or four objects at once even when they are occluded, so they are surprised when one disappears or appears without explanation.

Subtracting a currently elapsing duration from a remembered initial duration and comparing the difference to a standard duration (also drawn from memory) has been demonstrated in the 'time-left' paradigm[45]. It may thus reasonably be expected with numbers as well, although we do not yet have a compelling demonstration of this.

Multiplying and dividing

Rate is number divided by time. One symbol-processing model of animal conditioning is based on the assumption that animals estimate the rates of reinforcement predicted by different conditioned stimuli through a process of matrix inversion. Matrix inversion involves a combination of additions,

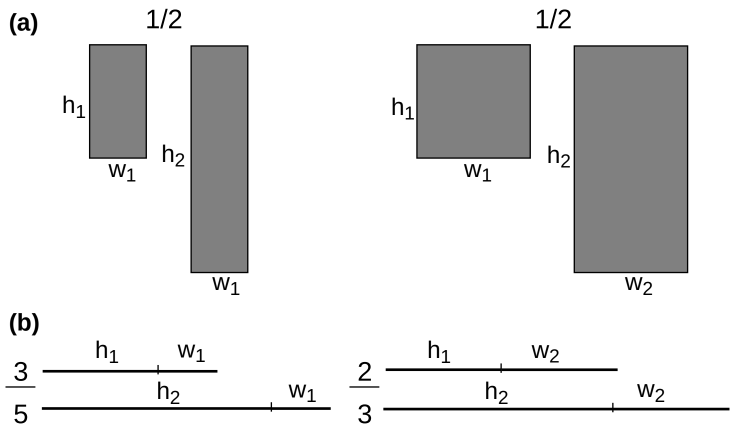

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Fig. 3. Contrasting properties of multiplicative and additive combination. (a) Height and width combine multiplicatively to determine area. Therefore, the effect of changing height (from h_1 to h_2) on the ratio of two areas is the same regardless of the width, that is, doubling height, doubles area, regardless of width. **(b)** By contrast, two lengths combine additively to determine a third length. Therefore, the effect of changing one of the two lengths on the ratio of the combinations depends on the other length. In the first example (left), doubling h makes the combination 5/3 as long as it was before. In the second example (right), doubling h makes the combination only 3/2 as long. The arithmetic nature of otherwise unknown combinatorial operations in the brain may be determined by looking at the behavioral effects of the combinatorial operation.

subtractions, multiplications and divisions[46]. Again, however, theory-dependent inferences of this sort are not compelling. Leon and Gallistel directly addressed the question of whether the subjective magnitudes of the brain-stimulation rewards that a rat received combined multiplicatively with the subjective rates of reward to determine the rat's preference in a free-operant matching paradigm[47]. The measure of preference was the ratio of the amount of time spent pressing one lever to the amount of time spent pressing the other. The preference was jointly determined by the rates of reinforcement, that is, by the numbers of rewards per unit of time (number/time) and by the magnitudes of the rewards. The magnitude of a reward was determined by the number of pulses (duration 0.1 ms) in the trains of electrical stimulation (duration 0.5 s) that reinforced lever-pressing. Increasing the rate of reward on one lever increased preference for that lever, as did increasing

Outstanding questions

- Are numerosities less than five apprehended by a noncounting process (subitizing)?[23,24,29,30,48,49]
- Are small numerosities represented discretely in the minds of infants (and other animals?), rather than by the mental magnitudes that represent larger numerosities?[4,24,32] If so, are these discrete representations of small numerosities arithmetically processed?
- Is the development of numerical cognition in humans rooted in the discrete representation of small numerosities or in a magnitude-based system for representing numerosities of any size, or both?[4,24,50,51]
- Are infants representing numerosity or just the overall extent or amount of a stimulus?[52,53]
- If even the smallest numerosities are represented by magnitudes with scalar variability, then why do the integers seem to be the foundation of numerical thinking?
- Does learning to use the verbal count list benefit from the discrete non-verbal counting process?[4,24,31,32,51,54]
- If humans represent numerosities in terms of magnitudes, why do they have so much trouble learning the mathematical conception of rational numbers (mastering fractions)?[55]

the number of pulses in the reinforcing trains delivered by that lever.

Leon and Gallistel showed that the factor by which preference increased in response to a change in the relative rates of reward was independent of the relative magnitudes of those rewards. Changing the relative rate of reward from, say, 1:1 to 10:1, increased preference about tenfold, regardless of whether all preferences were strongly biased towards one lever or towards the other by the differences in the magnitudes of rewards obtained from the two levers.

The Leon and Gallistel experiment is an example of how we may determine the arithmetic properties of an unknown combinatorial operation in the mind or brain by way of the behavioral effects of this process (see Fig. 3). Consider a variable M (for magnitude) whose effect on the brain combines (by some unknown combinatorial process) with the effects of a variable R (for rate) to produce some measurable behavioral effect, P (for preference). Manipulating the levels of either M or R produces different amounts of P. Symbolically, we may write:

$$P_{1,1} = f(M_1,R_1),$$

for the effect of level 1 of M combined with level 1 of R;

$$P_{1,2} = f(M_1,R_2);$$
$$P_{2,1} = f(M_2,R_1);$$

and so on, for the amounts of P produced by other combinations of M and R. The symbolism indicates that the observed preference depends on (is a function of) the measured levels of the variables M and R. If the individual effects of M and R on the brain combine multiplicatively within the brain, then the level of M will have no effect on the *ratio* of the two different preferences produced by different values of R. Thus, $P_{1,1}/P_{1,2}$, the ratio of the observed preferences when we try first R_1 and then R_2 while keeping M at level 1, is equal to the ratio $P_{2,1}/P_{2,2}$, which is the observed preferences when we choose a different level for M and again vary R from R_1 to R_2.

The level of M will not matter to the change in preference produced by changing R if and only if the effects (in the brain) of M and R combine multiplicatively, because then the effect of M will behave like a scaling factor with respect to R (and vice versa). Scaling factors cancel out of ratios (that is, $MR_1/MR_2 = R_1/R_2$). All other (mathematically distinguishable) forms of combination between the effects of M and R lack this property. For example, if the effects (in the brain) of M and R combine additively, then it will not be true that $P_{1,1}/P_{1,2} = P_{2,1}/P_{2,2}$, because additive factors do not cancel out of ratios. Thus, from a formalist perspective at least, an otherwise unknown and unobservable combinatorial process that exhibits this property is a form of multiplication. That is why Leon and Gallistel's results imply that subjective rates (the effects in the brain of the rates at which trains are delivered by the two different levers) combine multiplicatively with subjective magnitudes.

Conclusion

The results of current research on comparative numerical cognition speak to some of the oldest issues in science – the epistemological foundations of the numerical reasoning that permeates both scientific inquiry and everyday thought.

Although many have thought that numerical reasoning depends on language, this research suggests that it might not. It does not appear to be unique to humans; nor does it appear to be rooted in a system designed uniquely for the counting numbers (integers). Arithmetic reasoning is found in non-verbal animals, where it operates with real numbers (magnitudes). It may be that evolution provided the real numbers and that getting from integers back to the real numbers has been the work of man.

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